# A Hybrid Nonparametric Framework for Outlier Detection in Functional Time Series

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## 1. Motivation

Functional time series (e.g., climate, finance, epidemiology) consist of curve-valued observations rather than scalars. They are often contaminated by heterogeneous anomalies:

- Magnitude shifts: global level changes
- Shape distortions: structural changes in curves
- Local contamination: short-term disruptions

Existing IID-based detectors fail under temporal dependence, and most methods are limited to either global, local, or model-specific assumptions.

Goal: A unified, fully nonparametric method for detecting magnitude, shape, and partial outliers in dependent functional data.

## 2. Methods

We propose a hybrid framework combining global and local detection:

$$\text{DirOut} + \text{MBBo (global)} + \text{SWOD (local)}$$

The approach decomposes detection into:

- Global structure: detects overall functional deviations
- Local structure: captures transient or localized anomalies
- Dependence adjustment: resampling-based calibration under temporal dependence

### 2.1 Functional Depths

Functional depth provides a center-outward ordering of curves in functional space, extending the notion of multivariate data depth.

Given a sample

$$\{\chi_1, \dots, \chi_n\} \subset L^2(I),$$

a depth function assigns each curve a centrality score:

$$D(\chi_i; \{\chi_1, \dots, \chi_n\})$$

Large depth values indicate central observations; small values suggest potential outliers.

Depths used in this work

- Modified Band Depth (MBD)
  Measures how often a curve lies inside bands generated by pairs of sample curves.
- Modal Depth (MD)
  Measures local density around each curve.
- Multivariate MBD (multiMBD)
  Applies depth jointly to $(\chi_i, \chi_i', \chi_i'')$ to capture shape through derivatives.

### 2.2 MBBo and Raña et al. (2015)

Moving Block Bootstrap (MBB)

Temporal dependence invalidates standard bootstrap procedures. MBB preserves dependence by resampling overlapping blocks.

Given block length $l$:

$$B_i = (\chi_i, \dots, \chi_{i+l-1})$$

Bootstrap samples are built by sampling blocks with replacement.

Raña et al. (2015) procedure

The original MBBo-based functional outlier detection proceeds as:

1. Apply functional boxplot (Sun and Genton, 2011) to obtain clean subset $S_{\text{clean}}$
2. Generate $B$ bootstrap samples using MBB.
3. Compute depth values for each bootstrap replicate.
4. Estimate empirical cutoff threshold $C_\alpha$
5. Classify $\chi_i$ as outlier if $D(\chi_i) < C_\alpha$

### 2.3 DirOut: Directional Outlyingness

Directional Outlyingness (Dai & Genton, 2019) characterizes both magnitude and shape deviations.

For each time point $t$:

$$O(\chi(t), F_t) = \left(\frac{1}{d(\chi(t), F_t)} - 1\right) v(t)$$

where $d(\cdot)$ is a pointwise depth and $v(t)$ a directional unit vector.

Integrated summaries:

$$MO = \int O(t)\,dt \qquad VO = \int \|O(t) - MO\|^2\,dt \qquad FO = \|MO\|^2 + VO$$

Interpretation. $MO$ captures global magnitude deviation, $VO$ quantifies shape variability, and $FO$ summarizes total functional outlyingness.

Original test-based detection. DirOut performs outlier detection using a Gaussian approximation of directional outlyingness features and a robust Mahalanobis-distance F-test.

Limitation and contribution. Because this asymptotic $F$-calibration relies on independence, it may fail under temporal dependence. We therefore replace the theoretical cutoff $c_\alpha$ with an empirical threshold estimated via MBBo.

### 2.4 SWOD: Sliding Window Outlier Detection

SWOD detects local contextual anomalies using repeated window-based functional boxplots.
For each observation $\chi_i$, define

$$W_i = \{\chi_j : j = \max(1, i - \lfloor\omega/2\rfloor), \dots, \min(n, i + \lfloor\omega/2\rfloor)\}$$

Algorithm

1. Construct symmetric sliding window $W_i$
2. Compute functional depth within $W_i$
3. Apply functional boxplot (Sun and Genton, 2011) to identify local outliers
4. Record how often each curve is flagged
5. Declare global outlier if $\dfrac{\#\ \text{detections}}{\#\ \text{appearances}} > \tau$

Hyperparameters. The window size $\omega$ controls locality and is recommended between $\omega \approx [5\%, 8\%]$ of the sample size, while the persistence threshold is fixed at $\tau = 0.5$.

SWOD is fully nonparametric, and particularly effective for detecting local magnitude anomalies.

## 3. Algorithm

1. Compute DirOut measures
2. Calibrate thresholds using MBBo
3. Detect global shape/partial anomalies
4. Apply SWOD over sliding windows
5. Detect local magnitude anomalies
6. Combine outputs

## 4. Simulation Study

Functional time series were generated following Raña et al. (1025) using the AR(1) model

$$\zeta_i(t) = \cos(\pi t)(1 - \rho) + \rho\zeta_{i-1}(t) + a_i(t) + b_i,$$

with $n = 200$, $\rho = 0.8$, and 100 Monte Carlo replications. Three contamination mechanisms were considered:

- Magnitude outliers: global amplitude shifts
- Shape outliers: geometric perturbations
- Partial outliers: local deviations over subintervals

We compared the original MBBo approach, functional Boxplots, DirOut, and the proposed hybrid framework.

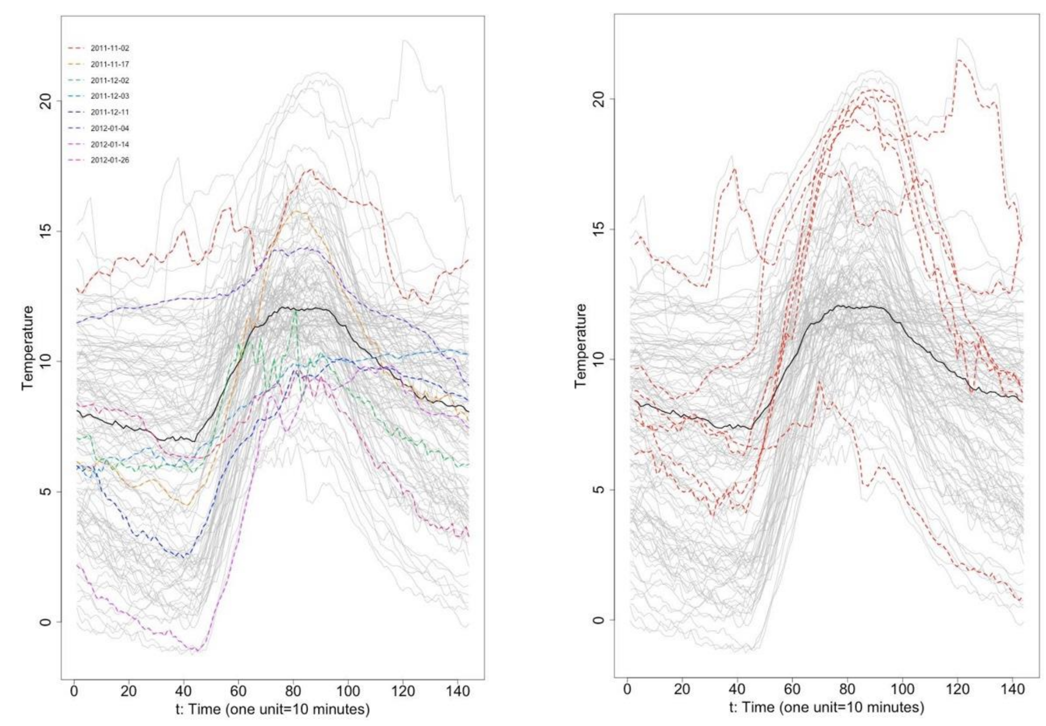
Main findings.

| Method | TPR | FPR |
|---|---|---|
| SWOD (Magnitude) | 84–100% | 0.5% |
| DirOut+MBBo (Shape) | 100% | 1.6% |
| DirOut+MBBo (Partial) | 100% | <1.5% |

Results show a clear complementarity: SWOD excels for magnitude anomalies, while DirOut+MBBo achieves near-perfect detection of shape and partial outliers.

## 5. Real Data Application

We analyzed daily temperature curves from Santiago de Compostela (Spain), recorded every 10 minutes from November 2011 to February 2012 ($n = 121$ daily functional observations).




(a) Additional outliers detected only by our framework

(b) Outliers jointly detected by previous and proposed method

The proposed hybrid method successfully recovered nearly all previously reported anomalies while identifying several additional curves with clearly atypical temperature patterns, demonstrating improved sensitivity to subtle functional deviations under temporal dependence.

## 6. References

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